

Multiplicativity of the maximal output 2-norm for depolarized Werner-Holevo channels.

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We study the multiplicativity of the output 2-norm for depolarized Werner-Holevo channels and show that multiplicativity holds for a product of two identical channels in this class. Moreover, it is shown that the depolarized Werner-Holevo channels do not satisfy the entrywise positivity condition introduced by C. King and M.B. Ruskai, which suggests that the main result is non-trivial.

THE SETUP AND MAIN RESULT

The d -dimensional Werner-Holevo channel $\mathcal{W}_d(\rho) = \frac{1}{d-1}((\text{Tr}(\rho))\mathbf{1}_d - \rho^T)$ is known [1] to give a counterexample to the multiplicativity of the maximal output p -norm for $p > 4.79$, when $d = 3$. Nevertheless, it has been shown [2, 3] that $\mathcal{W}_d(\rho)$ satisfies multiplicativity for $1 \leq p \leq 2$. It is natural then to study the output p -norm of depolarized Werner-Holevo channels

$$\mathcal{W}_{\lambda,d}(\rho) = \lambda\rho + (1-\lambda)\mathcal{W}_d(\rho),$$

and ask if those channels satisfy multiplicativity for p -norms with $p \leq 2$.

We focus our attention to the study of the output 2-norm for the tensor product channel $\mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}$ acting on bipartite states in $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ and show that multiplicativity is satisfied for this norm.

A direct computation of the eigenvalues of $\mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}(|\psi_{12}\rangle\langle\psi_{12}|)$ turns out to be much harder for $0 < \lambda < 1$, than for the boundary cases $\lambda = 0, 1$. The reason is that the output consists of a combination of the input state and its transpose/partial transpose, which in general do not share a common eigenbasis. To work around this difficulty, we compute explicitly the output 2-norm of $\mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}(|\psi_{12}\rangle\langle\psi_{12}|)$ and the maximal output 2-norm of $\mathcal{W}_{\lambda,d}$ and study the difference

$$\mathcal{D}_{\lambda,d}(\psi_{12}) = (\|\mathcal{W}_{\lambda,d}\|_2^2)^2 - \|\mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}(|\psi_{12}\rangle\langle\psi_{12}|)\|_2^2 \quad (1)$$

We show that $\mathcal{D}_{\lambda,d} \geq 0$ for all input states and $\lambda \in [0, 1]$, $d \geq 2$. We begin with the computation of $\|\mathcal{W}_{\lambda,d}\|_2^2$ in the following Lemma.

Lemma 1. *The (squared) maximal output 2-norm of $\mathcal{W}_{\lambda,d}$ is given by*

$$\|\mathcal{W}_{\lambda,d}\|_2^2 = \frac{(d-2)\lambda^2 + 1}{d-1}$$

Proof. It is easy to check that $\|\mathcal{W}_{\lambda,d}(|\psi\rangle\langle\psi|)\|_2^2$ is

$$\begin{aligned} &= \text{Tr}(\mathcal{W}_{\lambda,d}(|\psi\rangle\langle\psi|)^2) \\ &= \lambda^2 + \frac{2\lambda(1-\lambda)(1-|\langle\psi|\bar{\psi}\rangle|^2)}{d-1} + \frac{(1-\lambda)^2}{d-1} \\ &\leq \frac{(d-2)\lambda^2 + 1}{d-1}, \end{aligned}$$

where $|\bar{\psi}\rangle$ denotes the complex conjugate of $|\psi\rangle$ in the standard basis. Taking $|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$, with $|0\rangle, |1\rangle$ two standard basis vectors, we see that equality can be achieved in the above expression and the result follows. \square

We now turn our attention to the more complicated output 2-norm of $\mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}(|\psi_{12}\rangle\langle\psi_{12}|)$.

Lemma 2. *The (squared) output 2-norm $\|\mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}(|\psi_{12}\rangle\langle\psi_{12}|)\|_2^2$ is given by:*

$$\begin{aligned} &(\|\mathcal{W}_{\lambda,d}\|_2^2)^2 \\ &+ S_\lambda^2 |\langle\psi_{12}|\bar{\psi}_{12}\rangle|^2 \\ &- 2(S_\lambda + R_\lambda^2)(S_\lambda + (d-2)Q_\lambda^2)(1 - \|\rho_1\|_2^2) \\ &- S_\lambda \|\mathcal{W}_{\lambda,d}\|_2^2 \text{Tr}(\rho_1 \rho_1^T + \rho_2 \rho_2^T), \end{aligned}$$

where $Q_\lambda = \frac{1-\lambda}{d-1}$, $R_\lambda = \lambda - Q_\lambda$, $S_\lambda = 2\lambda Q_\lambda$, $\rho_1 = \text{Tr}_2|\psi_{12}\rangle\langle\psi_{12}|$, $\rho_2 = \text{Tr}_1|\psi_{12}\rangle\langle\psi_{12}|$ and T denotes transposition.

Proof. It is easy to check that

$$\begin{aligned} \mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}(|\psi_{12}\rangle\langle\psi_{12}|) &= \lambda^2 |\psi_{12}\rangle\langle\psi_{12}| \\ &+ Q_\lambda R_\lambda [\rho_1 \otimes \mathbf{1}_d + \mathbf{1}_d \otimes \rho_2] \\ &+ Q_\lambda^2 [\mathbf{1}_d \otimes \mathbf{1}_d + |\bar{\psi}_{12}\rangle\langle\bar{\psi}_{12}|] \\ &- \frac{S_\lambda}{2} (|\psi_{12}\rangle\langle\psi_{12}|)^{T_1} \\ &- \frac{S_\lambda}{2} (|\psi_{12}\rangle\langle\psi_{12}|)^{T_2}, \end{aligned}$$

where T_1, T_2 denote partial transposition w.r.t. the 1st, 2nd tensor factor, respectively. Taking the trace after squaring the above expression and noting that $\text{Tr}|\psi_{12}\rangle\langle\psi_{12}|(|\psi_{12}\rangle\langle\psi_{12}|)^{T_k} = \text{Tr}\rho_k \rho_k^T$, for $k = 1, 2$ (which one can show using the Schmidt decomposition of $|\psi_{12}\rangle$), we get the desired result. \square

The following general inequality will be very useful in the proof of the main theorem, so we state it here as a lemma.

Lemma 3. *Let $\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_d$ be non-negative numbers that sum up to 1. Then, the following inequality holds:*

$$\sigma_d \geq \sum_{\alpha=1}^d \sigma_\alpha^2$$

Proof. The r.h.s. of the inequality can be thought of as the expected value of the random variable X given by $\Pr(X = \sigma_\alpha) = \sigma_\alpha$. The upper bound then follows immediately. \square

PROOF OF THE MAIN RESULT

In this section, we will show that the difference $\mathcal{D}_{\lambda,d}$ defined in (1) is always non-negative, which is equivalent to multiplicativity of the output 2-norm for $\mathcal{W}_{\lambda,d}$. We state this as a theorem:

Theorem 4. *For the depolarized Werner-Holevo channel $\mathcal{W}_{\lambda,d}$, we have for $\lambda \in [0, 1]$, $d \geq 2$:*

$$\|\mathcal{W}_{\lambda,d} \otimes \mathcal{W}_{\lambda,d}\|_2 = \|\mathcal{W}_{\lambda,d}\|_2^2$$

Proof. From Lemma 2 we see that the condition $\mathcal{D}_{\lambda,d}(|\psi_{12}\rangle) \geq 0$ is equivalent to

$$\begin{aligned} S_\lambda^2 |\langle \psi_{12} | \overline{\psi_{12}} \rangle|^2 &\leq 2(S_\lambda^2 + P_\lambda^2)(1 - \|\rho_1\|_2^2) \\ &\quad + S_\lambda \|\mathcal{W}_{\lambda,d}\|_2^2 \text{Tr}(\rho_1 \rho_1^T + \rho_2 \rho_2^T), \end{aligned}$$

where $P_\lambda^2 = [Q_\lambda^2 + (d-2)R_\lambda^2]S_\lambda + (d-2)Q_\lambda^2 R_\lambda^2 \geq 0$. Using Lemma 1 to write $\|\mathcal{W}_{\lambda,d}\|_2^2$ as $(1 + \sqrt{d-1})S_\lambda + (\lambda - \frac{1-\lambda}{\sqrt{d-1}})^2$, we see that it is sufficient to prove the following inequality

$$|\langle \psi_{12} | \overline{\psi_{12}} \rangle|^2 \leq 2(1 - \|\rho_1\|_2^2) + (1 + \sqrt{d-1}) \text{Tr}(\rho_1 \rho_1^T + \rho_2 \rho_2^T) \quad (2)$$

(the boundary cases $\lambda = 0, 1$ follow from $1 \geq \|\rho_1\|_2^2$).

We will now make use of the Schmidt decomposition of the input state $|\psi_{12}\rangle\langle\psi_{12}|$, given by $|\psi_{12}\rangle = \sum_\alpha \sqrt{\sigma_\alpha} |\alpha_1\rangle \otimes |\alpha_2\rangle$, where $\{|\alpha_1\rangle\}, \{|\alpha_2\rangle\}$ are orthonormal sets in $M_d(\mathbb{C})$. We have that $\|\rho_1\|_2^2 = \sum_{\alpha=1}^d \sigma_\alpha^2$, where some of the σ_α may be zero. Applying Lemma 3 (and borrowing its notation w.l.o.g.), it follows that $\|\rho_1\|_2^2 \leq \sigma_d$. Moreover, it becomes clear now that in order to prove (2), it is sufficient to show:

$$|\langle \psi_{12} | \overline{\psi_{12}} \rangle|^2 \leq 2(1 - \sigma_d) + (1 + \sqrt{d-1}) \text{Tr}(\rho_1 \rho_1^T + \rho_2 \rho_2^T) \quad (3)$$

for $\sigma_d \geq 1/2$, since $|\langle \psi_{12} | \overline{\psi_{12}} \rangle| \leq 1$ and $\text{Tr}(\rho_1 \rho_1^T + \rho_2 \rho_2^T) \geq 0$. We now use the triangle inequality to get an estimate for the l.h.s. of (3),

$$\begin{aligned} |\langle \psi_{12} | \overline{\psi_{12}} \rangle| &= \left| \sum_{\alpha,\beta} \sqrt{\sigma_\alpha \sigma_\beta} \langle \alpha_1 | \overline{\beta_1} \rangle \langle \alpha_2 | \overline{\beta_2} \rangle \right| \\ &\leq \sum_{\alpha,\beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \quad (4) \end{aligned}$$

We will need to treat dimensions $d \leq 4$ and $d \geq 5$ separately. For $d \leq 4$ we use Cauchy-Schwarz to get the following estimate for

$$\left(\sum_{\alpha,\beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \right)^2$$

$$\begin{aligned} &\leq \left(\sum_{\alpha,\beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \overline{\beta_1} \rangle|^2 \right) \left(\sum_{\alpha,\beta} |\langle \alpha_2 | \overline{\beta_2} \rangle|^2 \right) \\ &\leq d \sum_{\alpha,\beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \overline{\beta_1} \rangle|^2 \quad (5) \end{aligned}$$

where we have used Parseval's identity in the last inequality. The same inequality is, of course, true for the second tensor factor. Using (3) and (4) along with the fact that

$$\text{Tr}(\rho_k \rho_k^T) = \sum_{\alpha,\beta} \sigma_\alpha \sigma_\beta |\langle \alpha_k | \overline{\beta_k} \rangle|^2 \quad k = 1, 2 \quad (6)$$

we see from estimate (5) that it is sufficient to show that $d \leq 2(1 + \sqrt{d-1})$, which is true for $d \leq 4$. We now turn our attention to the case $d \geq 5$. We will need a different estimate than the one given in (5), since we need to make use of the assumption that $\sigma_d \geq 1/2$ in order to lower the factor d in (5). We start by using Cauchy-Schwarz to get the following upper bound:

$$\left(\sum_{\alpha,\beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \right)^2 \leq 3(I_1^2 + I_2^2 + I_3^2), \quad (7)$$

where

$$\begin{aligned} I_1 &= \sum_{\alpha=d,\beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \\ I_2 &= \sum_{\alpha \neq d, \beta=d} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \\ I_3 &= \sum_{\alpha \neq d, \beta \neq d} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \end{aligned}$$

A further application of Cauchy-Schwarz on I_1, I_2, I_3 will give us the desired result. We start with an estimate for I_1 , since I_2 is very similar. Noting that one of the summation indices is fixed to d , we get

$$\begin{aligned} I_1^2 &= \left(\sum_{\alpha=d,\beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \right)^2 \\ &\leq \left(\sum_{\alpha=d,\beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \overline{\beta_1} \rangle|^2 \right) \left(\sum_{\alpha=d,\beta} |\langle \alpha_2 | \overline{\beta_2} \rangle|^2 \right) \\ &\leq \sum_{\alpha=d,\beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \overline{\beta_1} \rangle|^2 \\ &\leq \text{Tr}(\rho_1 \rho_1^T) \end{aligned}$$

Similarly, we see that $I_2^2 \leq \text{Tr}(\rho_2 \rho_2^T)$. Since $1 + \sqrt{d-1} \geq 3$ for $d \geq 5$, we see from (3) and (7) that it remains to show $3I_3^2 \leq 2(1 - \sigma_d)$. We have

$$\begin{aligned} I_3^2 &= \left(\sum_{\alpha \neq d, \beta \neq d} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \overline{\beta_1} \rangle| |\langle \alpha_2 | \overline{\beta_2} \rangle| \right)^2 \\ &\leq \left(\sum_{\alpha \neq d, \beta \neq d} \sigma_\alpha |\langle \alpha_1 | \overline{\beta_1} \rangle|^2 \right) \left(\sum_{\alpha \neq d, \beta \neq d} \sigma_\beta |\langle \alpha_2 | \overline{\beta_2} \rangle|^2 \right) \\ &\leq \left(\sum_{\alpha \neq d} \sigma_\alpha \right)^2 \\ &= (1 - \sigma_d)^2 \end{aligned}$$

It remains to show that $3(1 - \sigma_d)^2 \leq 2(1 - \sigma_d) \Leftrightarrow (1 - \sigma_d)(3\sigma_d - 1) \geq 0$, which follows from our assumption that $\sigma_d \in [\frac{1}{2}, 1]$. \square

DISCUSSION

We have shown that for depolarized Werner-Holevo channels the maximum output 2-norm is multiplicative. For $\lambda \in (0, 1)$ and $d \geq 3$, the depolarized Werner-Holevo maps do not satisfy the entrywise-positivity (EP) condition introduced by C. King and M.B. Ruskai in [4, 5]. This suggests that some elements of the above proof may be useful when tackling the multiplicativity of the maximal output 2-norm for arbitrary channels.

Proposition 5. *The depolarized Werner-Holevo channels $\mathcal{W}_{\lambda,d}$ with $\lambda \in (0, 1)$, $d \geq 3$ do not satisfy the entrywise-positivity (EP) condition:*

$$\text{Tr } \mathcal{W}_{\lambda,d}(|e_l\rangle\langle e_l|)\mathcal{W}_{\lambda,d}(|e_j\rangle\langle e_k|) \geq 0, \quad \forall i, j, k, l,$$

and $\{|e_i\rangle\}_{i=1}^d$ some orthonormal basis of \mathbb{C}^d .

Proof. One can check that $\text{Tr } \mathcal{W}_{\lambda,d}(|e_l\rangle\langle e_l|)\mathcal{W}_{\lambda,d}(|e_j\rangle\langle e_k|)$ is given by:

$$\begin{aligned} & \left[\lambda^2 + \left(\frac{1-\lambda}{d-1} \right)^2 \right] \delta_{i,j} \delta_{k,l} \\ & + \left[\frac{2\lambda(1-\lambda)}{d-1} + (d-2) \left(\frac{1-\lambda}{d-1} \right)^2 \right] \delta_{i,l} \delta_{j,k} \\ & - \frac{2\lambda(1-\lambda)}{d-1} \langle e_i | \overline{e_k} \rangle \langle \overline{e_l} | e_j \rangle \end{aligned}$$

where $|\overline{e_k}\rangle$ denotes the complex conjugate of $|e_k\rangle$, as before. Now, taking $i = j, k \neq l$ in the above expression, we see that the EP condition implies:

$$\langle e_i | \overline{e_k} \rangle \langle \overline{e_l} | e_i \rangle \leq 0, \quad \forall i, k \neq l.$$

Summing over i in the above inequality gives us 0, which implies that:

$$\langle e_i | \overline{e_k} \rangle \langle \overline{e_l} | e_i \rangle = 0, \quad \forall i, k \neq l. \quad (8)$$

Fixing l , we choose $i = \pi(l)$ such that $\langle \overline{e_l} | e_{\pi(l)} \rangle \neq 0$ (we can always find such a $\pi(l)$, since otherwise $|\overline{e_l}\rangle = 0$; a contradiction to $|\overline{e_l}\rangle$ being an orthonormal basis vector).

Condition (8) then implies that $\langle e_{\pi(l)} | \overline{e_k} \rangle = 0, \forall k \neq l$. Since the $\{|\overline{e_k}\rangle\}$ form an orthonormal basis, it follows that $|e_{\pi(l)}\rangle = |\overline{e_l}\rangle, \forall l$. We may now rewrite the EP condition as:

$$\begin{aligned} & \left[\lambda^2 + \left(\frac{1-\lambda}{d-1} \right)^2 \right] \delta_{i,j} \delta_{k,l} \\ & + \left[\frac{2\lambda(1-\lambda)}{d-1} + (d-2) \left(\frac{1-\lambda}{d-1} \right)^2 \right] \delta_{i,l} \delta_{j,k} \\ & \geq \frac{2\lambda(1-\lambda)}{d-1} \delta_{i,\pi(k)} \delta_{j,\pi(l)} \end{aligned}$$

Choosing $i = \pi(k), j = \pi(l)$ and $k \neq l$, the above condition becomes:

$$\left[\frac{2\lambda(1-\lambda)}{d-1} + (d-2) \left(\frac{1-\lambda}{d-1} \right)^2 \right] \delta_{\pi(k),l} \delta_{\pi(l),k} \geq \frac{2\lambda(1-\lambda)}{d-1}$$

The EP condition forces $\pi(l) = k, \forall k \neq l$ (note that $\pi(k) = l$ then follows from the definition of $\pi(k)$), which is impossible for $d \geq 3$. For $d = 2$, choosing $|e_1\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ satisfies the EP condition. \square

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